ON A FUZZY COMPLETELY CLOSED IDEAL
OF A BH-ALGEBRA

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Abstract
In this paper, we introduce the notion of a fuzzy completely closed ideal of a BH-algebra and study this notion on a BG-algebra. Also We stated and prove some theorems which determine the relationships between this notion and some types of fuzzy ideals of a BH-algebra.

INTRODUCTION
The notion of BCK-algebra introduced by Y. Imai and K. Iseki in 1966[19]. In the same year, K. Iseki introduced the notions of a BCI-algebra as a generalization of a BCK-algebra and the notion of ideal of a BCK-algebra[9]. In 1983, Q.P.Hu and X.Li introduced the notion of a BCH-algebra which was a generalization of BCK/BCI-algebras [13]. In 1991, C. S. Hoo introduced the notion of an ideal, closed ideal and filter in a BCI-algebra[4]. In 1998, Y. B. Jun, E. H. Rogh and H. S. Kim introduced the notion of BH-algebra, which is a generalization of BCH-algebras[17]. In 2002, J. Neggers. and H. S. Kim introduced the notion

On the other hand, we will mention the development of a fuzzy set, fuzzy subalgebra, fuzzy ideals, fuzzy closed ideals and some other types of fuzzy ideals.


In this paper, we introduce the notions as we mentioned in the abstract.

1. PRELIMINARIES

In this section, we give some basic concepts about a BG-algebra, a BH-algebra, ideal of a BH-algebra, closed ideal of a BH-algebra, a completely closed ideal of a BH-algebra, closed ideal with respect to an element of a BH-algebra, completely closed ideal with respect to an element of a BH-algebra, a normal set, fuzzy set, fuzzy ideal, fuzzy closed ideal, fuzzy closed ideal with respect to an element of a BH algebra with some theorems and propositions which we needed in our work.

**Definition (1.1) [2]:**

A BG-algebra is a non-empty set $X$ with a constant 0 and a binary operation “*” satisfying the following axioms:
1) $x * x = 0$,
2) $x * 0 = x$,
3) $(x * y) * (0 * y) = x$, for all $x, y \in X$.

**Definition (1.2) [17]:**

A BH-algebra is a nonempty set $X$ with a constant 0 and a binary operation * satisfying the following conditions:
1) \( x \ast x = 0, \ \forall x \in X. \)
2) \( x \ast y = 0 \) and \( y \ast x = 0 \) imply \( x = y, \ \forall x, y \in X. \)
3) \( x \ast 0 = x, \ \forall x \in X. \)

**Proposition (1.3) [2]:**
Every BG-algebra is a BH-algebra.

**Definition (1.4) [14]:**
A nonempty subset \( S \) of a BH-algebra \( X \) is called a BH-Subalgebra or Subalgebra of \( X \) if \( x \ast y \in S \) for all \( x, y \in S \).

**Definition (1.5) [17]:**
Let \( I \) be a nonempty subset of a BH-algebra \( X \). Then \( I \) is called an ideal of \( X \) if it satisfies:
1) \( 0 \in I. \)
2) \( x \ast y \in I \) and \( y \in I \) imply \( x \in I \).

**Definition (1.6) [5]:**
An ideal \( I \) of a BH-algebra \( X \) is called a closed ideal of \( X \) if for every \( x \in I \), we have \( 0 \ast x \in I \).

**Definition (1.7) [7]:**
An ideal \( I \) of a BH-algebra is called a completely closed ideal if \( x \ast y \in I, \forall x, y \in I \).

**Definition (1.8) [10]:**
Let \( X \) be a non-empty set. A fuzzy set \( A \) in \( X \) (a fuzzy subset of \( X \)) is a function from \( X \) into the closed interval \([0,1]\) of the real number.

**Definition (1.9) [5]:**
Let \( A \) and \( B \) be two fuzzy sets in \( X \), then:
\[
(A \cap B)(x) = \min\{A(x), B(x)\}, \text{ for all } x \in X.
\]
\[
(A \cup B)(x) = \max\{A(x), B(x)\}, \text{ for all } x \in X.
\]
\( A \cap B \) and \( A \cup B \) are fuzzy sets in \( X \).

In general, if \( \{A_\alpha, \alpha \in \Lambda\} \) is a family of fuzzy sets in \( X \), then:
\[
\left( \bigcap_{i \in \Gamma} A_i \right)(x) = \inf\{A_i(x), i \in \Gamma\}, \text{ for all } x \in X \text{ and } 
\]
\[
\bigcup_{i \in I} A_i(x) = \sup \{ A_i(x), i \in I \}, \text{ for all } x \in X.
\]

which are also fuzzy sets in X.

**Definition (1.10) [11]:**

Let A be a fuzzy set in X, for all \( t \in [0,1] \). The set \( A_t = \{ x \in X, A(x) \geq t \} \) is called a level subset of A.

**Definition (1.11)[14]:**

A fuzzy set A in a BH-algebra X is said to be a fuzzy subalgebra of X if it satisfies:
\[
A(x \ast y) \geq \min \{ A(x), A(y) \}, \ \forall x, y \in X.
\]

**Remark (1.12)[3]:**

A fuzzy subset A of a BH-algebra X is said to be a fuzzy ideal if and only if:
1) For any \( x \in X \), \( A(0) \geq A(x) \).
2) For any \( x, y \in X \), \( A(x) \geq \min \{ A(x \ast y), A(y) \} \).

**Definition (1.13) [6]:**

A fuzzy ideal A of a BH-algebra X is said to be closed if
\[
A(0 \ast x) \geq A(x) \quad \text{for any } x \in X.
\]

**Theorem (1.14)[6]:**

A fuzzy set A of a BH-algebra X is called a fuzzy p-ideal of X if it satisfies:
1) \( A(0) \geq A(x) \), For any \( x \in X \).
2) \( A(x) \geq \min \{ A((x \ast z) \ast (y \ast z)), A(y) \} \), for all \( x, y, z \in X \).

**Definition (1.15)[6]:**

A fuzzy set A of a BH-algebra X is called a fuzzy a-ideal of X if it satisfies
i. \( A(0) \geq A(x) \), For any \( x \in X \).
ii. \( A(y \ast x) \geq \min \{ A((x \ast z) \ast (0 \ast y)), A(z) \} \), for all \( x, y, z \in X \).
Definition (1.16)[18]:
A fuzzy set M in a B-algebra X is said to be fuzzy normal if it satisfies
the inequality \( M((x*a)*(y*b)) \geq \min\{M(x*y),M(a*b)\} \), for all a, b, x, y \( \in \) X.

Lemma (1.17)[2]:
Let \((X,*,0)\) be a BG-algebra. Then
1) The right cancellation law holds in X, i.e., \( x*y = z*y \) implies \( x=z \),
2) \( 0*(0*x) = x \) for all \( x \in X \),
3) If \( x*y = 0 \), then \( x=y \) for any \( x,y \in X \),
4) If \( 0*x = 0*y \), then \( x=y \) for any \( x,y \in X \),
5) \((x*(0*x))*x = x \) for all \( x \in X \).

Proposition (1.18)[6]:
Let X be a BH-algebra. Then every fuzzy p-ideal of X is a fuzzy ideal of X.

Proposition (1.19)[7]:
A BH-algebra X is called an associative BH-algebra if:
\( (x*y)*z = x*(y*z) \), for all \( x,y,z \in X \).

Proposition (1.20)[7]:
Let X be an associative BH-algebra. Then the following properties are hold:
1) \( 0*x = x \) \( \forall x \in X \). 2) \( x*y = y*x \) \( \forall x,y \in X \).

2. THE MAIN RESULTS
In this section, we define the notion of a fuzzy completely closed ideal of a BH-algebra. For our discussion, we shall link this notion with other notions which mentioned in preliminaries.

Definition (2.1):
Let X be a BH-algebra and A be a fuzzy ideal of X. Then A is called a fuzzy completely closed ideal, if \( A(x*y) \geq \min\{A(x),A(y)\}, \forall x,y \in X \).
Example (2.2):
Consider the BH-algebra \( X = \{0, 1, 2, 3\} \) with the following operation table.

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The fuzzy ideal \( A \) which is defined by
\[
A(x) = \begin{cases} 
0.5 & x = 0, 1 \\
0.4 & x = 2, 3 
\end{cases}
\]
is a fuzzy completely closed ideal.

Theorem (2.3):
Let \( X \) be a BH-algebra. If \( x \cdot y = z, z \in \{0, x, y\} \ \forall x, y \in X \), then every fuzzy ideal is a fuzzy completely closed ideal.

**Proof:**
Let \( A \) be a fuzzy ideal and \( x, y \in X \).
- If \( x \cdot y = 0 \), \( \Rightarrow A(x \cdot y) = A(0) \geq \min\{A(x), A(y)\} \)
- If \( x \cdot y = x \), \( \Rightarrow A(x \cdot y) = A(x) \geq \min\{A(x), A(y)\} \)
- If \( x \cdot y = y \), \( \Rightarrow A(x \cdot y) = A(y) \geq \min\{A(x), A(y)\} \)

\( \therefore A \) is a fuzzy completely closed ideal.

Proposition (2.4):
Let \( X \) be an associative BH-algebra. Then every fuzzy ideal is a fuzzy closed ideal.

**Proof:**
Let \( A \) be a fuzzy ideal, and \( x \in X \).
\[
A(0 \cdot x) = A(x) \geq A(x) \quad [\text{By proposition (1.20)(1)}]
\]
\( \therefore A \) is a fuzzy closed ideal.
Theorem (2.5):
Let $X$ be a BH-algebra. If $A$ is a fuzzy completely closed ideal. Then $A_{\alpha}$ is a completely closed ideal for all $\alpha \in [0, A(0)]$.

Proof:
To prove $A_{\alpha}$ is an ideal,
1) Since $\alpha \in [0, A(0)] \Rightarrow A(0) \geq \alpha \quad \forall \alpha \in [0, A(0)]$.
   $\Rightarrow 0 \in A_{\alpha} \quad \forall \alpha \in [0, A(0)]$.
2) Let $x^*y \in A_{\alpha}, y \in A_{\alpha} \Rightarrow A(x^*y) \geq \alpha, A(y) \geq \alpha \Rightarrow \min \{A(x^*y), A(y)\} \geq \alpha$
But $A(x) \geq \min\{A(x^*y), A(y)\}$
   $\Rightarrow A(x) \geq \alpha \Rightarrow x \in A_{\alpha}$
:. $A_{\alpha}$ is an ideal.
Now, let $x, y \in A_{\alpha}$
$\Rightarrow A(x) \geq \alpha, A(y) \geq \alpha \Rightarrow \min \{A(x), A(y)\} \geq \alpha \Rightarrow A(x^*y) \geq \alpha \Rightarrow x^*y \in A_{\alpha}$
:. $A_{\alpha}$ is a completely closed ideal. ■

Proposition (2.6):
Let $X$ be an associative BH-algebra. If $A$ is a fuzzy $P$-ideal, then $A$ is a fuzzy closed ideal.

Proof:
Since $A$ is a fuzzy a $P$-ideal,
:. $A$ is a fuzzy ideal [By proposition(1.18)]
Now, let $x \in X$
$\Rightarrow A(0^*x) = A(x) \geq A(x)$ [By Proposition(1.20)(1)]
:. $A$ is a fuzzy closed ideal. ■

Theorem (2.7):
Let $X$ be BH-algebra. If $A$ is a fuzzy ideal, then the set $X_{\lambda}=\{x \in X: A(x)=A(0)\}$ is an ideal.

Proof:
Let $A$ be a fuzzy ideal.
1) Since $A(0)=A(0)$, $\Rightarrow 0 \in X_{\lambda}$
Theorem (2.8):
Let $X$ be BH-algebra and $A$ be a fuzzy completely closed ideal. Then the set $X_A = \{ x \in X : A(x) = A(0) \}$ is a completely closed ideal.

Proof:
Let $A$ be a fuzzy completely closed ideal.

$\Rightarrow$ $A$ is a fuzzy ideal, $\Rightarrow X_A$ is an ideal, $\quad [\text{By theorem (2.7)}]$

Now, let $x, y \in X_A$

$\Rightarrow A(x) = A(y) = A(0) \Rightarrow \min \{ A(x), A(y) \} = A(0)$
But $A(x*y) \geq \min \{ A(x), A(y) \} = A(0) \Rightarrow A(x*y) \geq A(0)$
But $A(0) \geq A(x*y) \Rightarrow A(x*y) = A(0)$

$\therefore x*y \in X_A$

$\therefore X_A$ is a completely closed ideal.$\blacksquare$

Theorem (2.9):
Let $X$ be BH-algebra and let $A$ be a fuzzy set. Then $A$ is a fuzzy ideal if and only if $A'(x) = A(x) + 1 - A(0)$ is a fuzzy ideal.

Proof:
Let $A$ be a fuzzy ideal,

1) $A'(0) = A(0) + 1 - A(0)$,

$\Rightarrow A'(0) = 1 \Rightarrow A'(0) \geq A'(x) \quad \forall x \in X$

2) $A'(x) = A(x) + 1 - A(0)$

$\geq \min \{ A(x*y), A(y) \} + 1 - A(0)$

$\geq \min \{ A(x*y) + 1 - A(0) , A(y) + 1 - A(0) \}$

$\geq \min \{ A'(x*y), A'(y) \}$
\[ \therefore A'(x) \geq \min\{A'(x \cdot y), A'(y)\} \]

\[ \therefore A' \text{ is a fuzzy ideal.} \]

**Conversely**

Let \( A' \) be a fuzzy ideal.

1) \( A(0) = A'(0) - 1 + A(0) \)

\[ \implies A(0) \geq A'(x) - 1 + A(0) \]

\[ \implies A(0) \geq A(x) \quad \forall x \in X \]

2) \( A(x) = A'(x) - 1 + A(0) \geq \min\{A'(x \cdot y), A'(y)\} - 1 + A(0) \)

\[ \geq \min\{A'(x \cdot y) - 1 + A(0), A'(y) - 1 + A(0)\} \]

\[ \geq \min\{A(x \cdot y), A(y)\} \]

\[ \therefore A(x) \geq \min\{A(x \cdot y), A(y)\} \]

\[ \therefore A \text{ is a fuzzy ideal.} \]

**Theorem (2.10):**

Let \( X \) be BH-algebra and \( A \) be a fuzzy set of \( X \). Then \( A \) is a fuzzy completely closed ideal if and only if \( A'(x) = A(x) + 1 - A(0) \) is a fuzzy completely closed ideal.

**Proof:**

Let \( A \) be a fuzzy completely closed ideal,

\[ \implies A \text{ is a fuzzy ideal.} \implies A' \text{ is a fuzzy ideal.} \quad \text{[By theorem(2.9)]} \]

Now,

Let \( x, y \in X \)

\[ A'(x \cdot y) = A(x \cdot y) + 1 - A(0) \]

\[ \geq \min\{A(x), A(y)\} + 1 - A(0) \]

\[ \geq \min\{A(x) + 1 - A(0), A(y) + 1 - A(0)\} \]

\[ \geq \min\{A'(x), A'(y)\} \]

\[ \therefore A'(x) \geq \min\{A'(x), A'(y)\} \]

\[ \therefore A' \text{ is a fuzzy completely closed ideal} \]

**Conversely**

Let \( A' \) be a fuzzy completely closed ideal,

\[ \implies A' \text{ is a fuzzy ideal.} \implies A \text{ is a fuzzy ideal.} \quad \text{[By theorem(2.9)]} \]

Now,

Let \( x, y \in X \)
\[ A(x*y)=A'(x*y)-1+A(0) \]
\[ \geq \min\{A'(x),A'(y)\}-1+A(0) \]
\[ \geq \min\{A'(x)-1+A(0),A'(y)-1+A(0)\} \]
\[ \geq \min\{A(x),A(y)\} \]
\[ \therefore A(x) \geq \min\{A(x),A(y)\} \]
\[ \therefore A' \text{ is a fuzzy completely closed ideal}. \]

**Proposition (2.11):**

Let \( X \) be a BH-algebra. Then every fuzzy normal set is a fuzzy subalgebra.

**Proof:**

Let \( M \) be a fuzzy normal set and \( x, y \in X \),
\[ M(x*y)=M((x*y)*(0*0)) \]
\[ \geq \min\{M(x*0),M(y*0)\} \]
\[ \geq \min\{M(x),M(y)\} \]
\[ \therefore M \text{ is a fuzzy subalgebra}. \]

**Proposition (2.12):**

Let \( X \) be a BH-algebra. If \( M \) is a fuzzy normal set, then \( M(0) \geq M(x) \) \( \forall x \in X \).

**Proof:**

Let \( M \) be a fuzzy normal set and \( x \in X \).
\[ M(0)=M((x*x)*(0*0)) \geq \min\{M(x*0),M(x*0)\} \]
\[ \geq \min\{M(x),M(x)\} \geq M(x) \]
\[ \therefore M(0) \geq M(x). \]

**Proposition (2.13):**

Let \( X \) be an associative BH-algebra. Then every fuzzy normal set is a fuzzy ideal.

**Proof:**

Let \( M \) be a fuzzy normal set.
1) \( M(0)=M((x*x)*(0*0)) \geq \min\{M(x*0),M(x*0)\} \)
\[ \geq \min\{M(x),M(x)\}\geq M(x) \]

2) \( M(x)=M(x*0) = M(x*(y*y)) = M((x*0)*(y*y)) \geq \min\{M(x*y),M(0*y)\} \)
\[ \geq \min\{M(x*y),M(y)\} \quad \text{[By proposition(1.20)(1)]} \]

\( \therefore M \) is a fuzzy ideal. ■

**Proposition (2.14):**

Let \( X \) be an associative BH-algebra. Then every fuzzy normal set is a fuzzy completely closed ideal.

**Proof:**

Let \( M \) be a fuzzy normal set.

\( \Rightarrow M \) is a fuzzy ideal \[ \text{[By proposition(2.13)]} \]

Now,

Let \( x, y \in X \),

\[ M(x*y)=M((x*y)*(0*0)) \geq \min\{M(x*0),M(y*0)\} \geq \min\{M(x),M(y)\} \]

\( \therefore \) \( M \) is a fuzzy completely closed ideal. ■

**Proposition(2.15):**

Let \( X \) be an associative BH-algebra. Then every fuzzy a-ideal is a fuzzy ideal.

**Proof:**

Let \( A \) be a fuzzy a-ideal,

1) \( A(0) \geq A(x), \forall x \in X \)

2) Let \( x, z \in X \)

\[ A(x)=A(x*0) \geq \min\{A(x*z)*(0*0),A(z)\} \]

Now, Let \( z=y \)

\[ \geq \min\{A(x*y)*(0*0),A(y)\} \geq \min\{A(x*y),A(y)\} \]

\( \therefore A \) is a fuzzy ideal. ■

**Theorem(2.16):**

Let \( X \) be an associative BH-algebra. Then every fuzzy a-ideal is a fuzzy completely closed ideal.
Proof:
Let $A$ be a fuzzy a-ideal,
$\Rightarrow A$ is a fuzzy ideal, [By Proposition(2.15)]

Now, let $x, y \in X$

$A(x*y) = A(y*x)$ [By Proposition(1.20)(2)]
$\geq \min\{A(x*z)*(0*y), A(z)\}$

Now, when $y = z$
$\geq \min\{A(x*y)*(0*y), A(y)\} \geq \min\{A(x), A(y)\}$

$\therefore A$ is a fuzzy completely closed ideal. ■

Proposition (2.17):
Let $\{A_i : i \in \Gamma\}$ be a family of fuzzy completely closed ideals of a BH-algebra $X$. Then $\bigcap_{i \in \Gamma} A_i$ is a fuzzy completely closed ideal of $X$.
Proof:

To prove that $\bigcap_{i \in \Gamma} A_i$ is a fuzzy ideal,

(1) Let $x \in X$.

$\left( \bigcap_{i \in \Gamma} A_i \right)(0) = \inf\{ A_i(0), i \in \Gamma \}$

$\geq \inf\{A_i(x), i \in \Gamma \}$

$= \left( \bigcap_{i \in \Gamma} A_i \right)(x)$

$\Rightarrow \left( \bigcap_{i \in \Gamma} A_i \right)(0) \geq \left( \bigcap_{i \in \Gamma} A_i \right)(x)$

(2) Let $x, y \in X$
\[
\left( \bigcap_{i \in \Gamma} A_i \right)(x) = \inf\{ A_i(x), \ i \in \Gamma \}
\]

\[
\geq \inf\{ \min\{ A_i(x*y), A_i(y) \}, \ i \in \Gamma \}
\]

\[
= \min\{ \inf\{ A_i(x*y), \ i \in \Gamma \}, \inf\{ A_i(y), \ i \in \Gamma \} \}
\]

\[
= \min\{ \left( \bigcap_{i \in \Gamma} A_i \right)(x*y), \left( \bigcap_{i \in \Gamma} A_i \right)(y) \}
\]

\[
\Rightarrow \left( \bigcap_{i \in \Gamma} A_i \right)(x) \geq \min\{ \left( \bigcap_{i \in \Gamma} A_i \right)(x*y), \left( \bigcap_{i \in \Gamma} A_i \right)(y) \}
\]

Therefore,

\[
\left( \bigcap_{i \in \Gamma} A_i \right)
\]

is a fuzzy ideal of X.

To prove that \(\left( \bigcap_{i \in \Gamma} A_i \right)\) is a fuzzy completely closed ideal of X

Let \(x, y \in X\)

\[
\left( \bigcap_{i \in \Gamma} A_i \right)(x*y) = \inf\{ A_i(x*y), \ i \in \Gamma \}
\]

\[
\geq \inf\{\min\{ A_i(x), A_i(y) \}, \ i \in \Gamma \}
\]

\[
\geq \min\{\inf\{ A_i(x), \ i \in \Gamma \}, \inf\{ A_i(y), \ i \in \Gamma \} \}
\]

\[
\geq \min\{ \left( \bigcap_{i \in \Gamma} A_i \right)(x), \left( \bigcap_{i \in \Gamma} A_i \right)(y) \}
\]

\[
\left( \bigcap_{i \in \Gamma} A_i \right) \qquad \left( \bigcap_{i \in \Gamma} A_i \right)_{13} \qquad \left( \bigcap_{i \in \Gamma} A_i \right)
\]
\[ (x \ast y) \geq \min \{ (x), (y) \} \quad \forall x, y \in X \]

Therefore, \( \bigcap_{i \in I} A_i \) is a fuzzy completely closed ideal of \( X \). ■

REFERENCES


حول المثالية الضبابية المغلقة تماما في جبر BH

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المستخلص
قدمنا في هذا البحث مفهوم المثالية الضبابية المغلقة تماما في جبر BH. كما درسنا هذا المفهوم في جبر BG. كما وضعنا وأثبتنا بعض المبرهنات ذات العلاقة بين هذا المفهوم و بعض أنواع المثاليات الضبابية في جبر BG و جبر BH.